3D Euler Equations on Manifolds with Symmetry

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Let M be a compact, oriented Riemannian manifold of dimension $n = 2, 3$. The movement of an ideal fluid filling M can be described by the Euler equations

$$\begin{aligned} &\partial_t u + \nabla_u u = -\nabla p \\ &\mathrm{div}(u) = 0 \\ &u(0) = u_0 \end{aligned} \tag{E}$$

where $u : M \times \mathbb{R} \to TM$ is the velocity and $p : M \times \mathbb{R} \to \mathbb{R}$ the pressure.

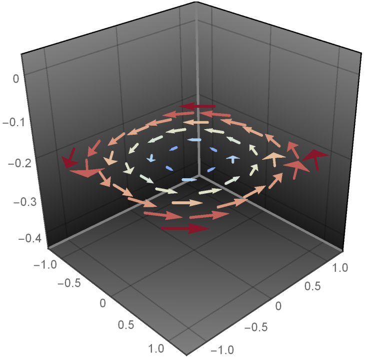
Remark 1

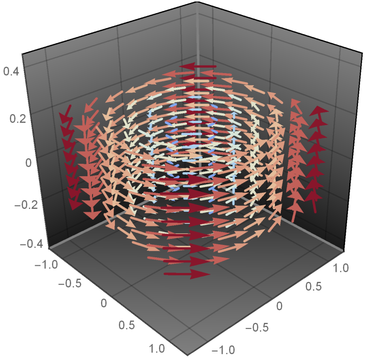
In Sobolev spaces H^s (for sufficiently high s), the problem (E) is locally well-posed in any dimension. It is globally well-posed in dimension two, but this is not known in dimension three.

Definition 1

A vector field u in $\mathbb{R}^3$ *is* axisymmetric *if u does not depend on* θ *in cylindrical coordinates, i.e.,*

$$u(r,z) = u_1(r,z)\partial_r + u_2(r,z)\partial_\theta + u_3(r,z)\partial_z. \tag{1}$$





Definition 1

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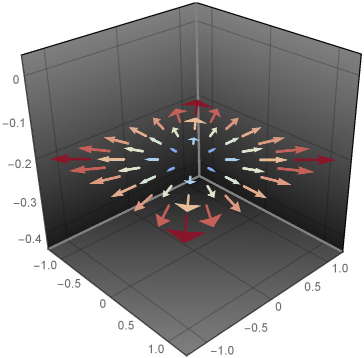
$$u(r,z) = u_1(r,z)\partial_r + u_2(r,z)\partial_\theta + u_3(r,z)\partial_z. \tag{1}$$

Definition 2

An axisymmetric vector field u is swirl-free *if it has no ∂_θ component, i.e.,*

$$u(r,z) = u_1(r,z)\partial_r + u_3(r,z)\partial_z. \tag{2}$$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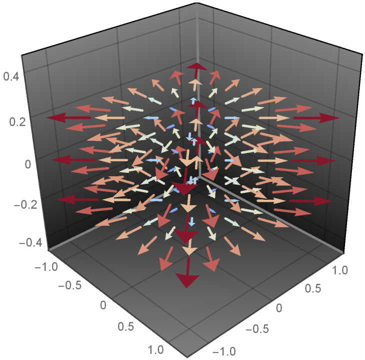
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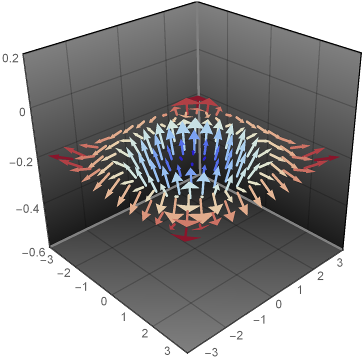
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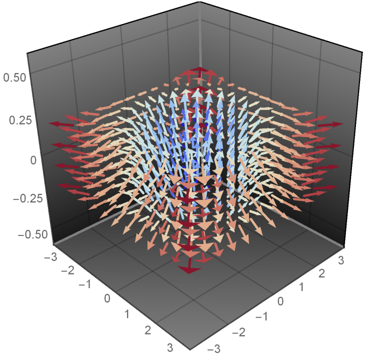
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Theorem (Ukhovskii, Yudovich, '68)

The 3D swirl-free Euler equations are globally well-posed.

Remark 2

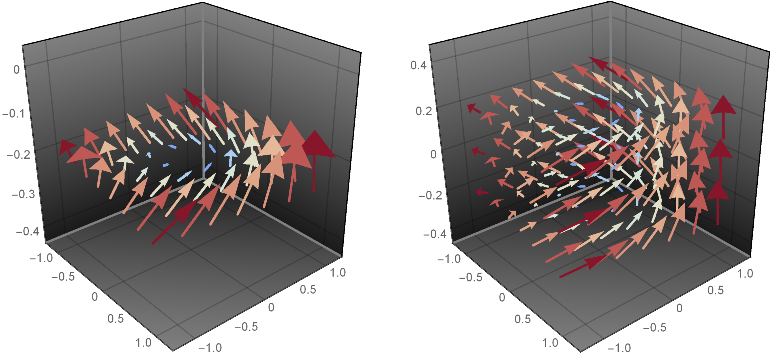
Global well-posedness is still unknown in general for the axisymmetric Euler equations with swirl*.*

Remark 3

Since in $\mathbb{R}^3$ infinitesimal isometries are generated by rotations and translations, the only other interesting case is that of helicoidal *symmetry.*

Definition 3

A vector field in $\mathbb{R}^3$ *is helicoidal if it commutes with the vector field* $\theta_\theta + \partial_z$.



Definition 4

A vector field in $\mathbb{R}^3$ *is helicoidal if it commutes with the vector field* $\partial_\theta + \partial_z$.

Definition 5

If u *is a helicoidal vector field, the swirl of* u *is the quantity*

$$\sigma = \langle u, \partial_\theta + \partial_z \rangle$$

Theorem 1 (Dutrifoy, '99)

The 3D helicoidal, swirl-free Euler equations are globally well-posed.

Let M be a Riemannian manifold and $K \neq 0$ a Killing field on M.

Definition 6

A vector field u on M is axisymmetric *if* $[u, K] = 0$.
If u is axisymmetric, its swirl *is the function* $\sigma = \langle u, K \rangle$.
When $\sigma \equiv 0$, *u is called* swirl-free.

Theorem 2 (L., Misiołek, Preston '18)

If u_0 is axisymmetric (resp. swirl-free), then the corresponding solution $u(t)$ of the Euler equations remains axisymmetric (resp. swirl-free), for as long as it exists.

A different way to model a fluid is to track the *position*, rather than the *velocity*, of each particle over time.

In this description, we let $\eta(x, t)$ be the **position, at time t, of the particle that started at x**.

By definition, $\eta(x, 0) = x$ for all x.

Since fluid particles are not allowed to fuse together or split, for each fixed time t, the map

$$\begin{aligned} \eta_t : M &\to M \\ x &\mapsto \eta(x, t) \end{aligned} \tag{3}$$

is a bijection. In fact, it is a volume-preserving diffeomorphism.

Therefore, $t \mapsto \eta_t$ is a curve in $\mathcal{D}_\mu(M)$, the group of Sobolev volume-preserving diffeomorphisms of M, starting at the identity map, $\eta_0 = e$.

For technical reasons, it is convenient to work with diffeomorphisms which are of Sobolev class, rather than C^∞ diffeomorphisms.

The group

$$\mathcal{D}^s_\mu(M) = \{\eta : M \to M \,:\, \eta, \eta^{-1} \in H^s(M, M), \text{ and } \eta^*\mu = \mu \,\}$$

of Sobolev volume-preserving diffeomorphisms is a C^∞ Hilbert manifold (*Eells* '66; *Ebin-Marsden* '70; *Omori* '74).

Its tangent space at the identity id is the set

$$T_e\mathcal{D}^s_\mu(M) = \{v \in H^s(TM) \,:\, \operatorname{div} v = 0\}$$

of all divergence-free vector fields on M. At other points $\eta \in \mathcal{D}^s_\mu(M)$, the tangent space is

$$T_\eta\mathcal{D}^s_\mu(M) = \left\{v \circ \eta \,:\, v \in T_e\mathcal{D}^s_\mu(M)\right\}$$

$\mathcal{D}^s_\mu(M)$ is not a Lie group: right multiplication

$$\begin{aligned} R_\eta : \mathcal{D}^s_\mu(M) &\to \mathcal{D}^s_\mu(M) \\ \xi &\mapsto \xi \circ \eta \end{aligned} \tag{4}$$

is smooth, since

$$\begin{aligned} dR_\eta(e) : T_e\mathcal{D}^s_\mu(M) &\to T_\eta\mathcal{D}^s_\mu(M) \\ u &\mapsto u \circ \eta \end{aligned} \tag{5}$$

However, *left* multiplication

$$\begin{aligned} L_\eta : \mathcal{D}^s_\mu(M) &\to \mathcal{D}^s_\mu(M) \\ \xi &\mapsto \eta \circ \xi \end{aligned} \tag{6}$$

is not smooth, because the expression

$$\begin{aligned} dL_\eta(e) : T_e\mathcal{D}^s_\mu(M) &\to T_\eta\mathcal{D}^{s-1}_\mu(M) \\ u &\mapsto D\eta \circ u \end{aligned} \tag{7}$$

causes a loss of one derivative due to the $D\eta$ term.

$\mathcal{D}^s_\mu(M)$ also carries a natural right-invariant Riemannian metric, given at the tangent space to the identity map by

$$\langle u, v \rangle_{L^2} = \int_M \langle u, v \rangle \, dV, \quad u, v \in T_e \mathcal{D}^s_\mu(M)$$

This is a *weak* Riemannian metric, i.e., it defines a topology (L^2) which is weaker than the topology our manifold has (H^s).

Therefore, the existence of a smooth Levi-Civita connection and geodesic spray are not guaranteed, but must be proved separately. This was done by Ebin and Marsden in their 1970s paper.

Arnold's insight: a curve

$$\eta : [0, T) \to \mathcal{D}^s_\mu(M), \quad \eta(0) = \mathrm{id},$$

is a geodesic ($\eta''(t) = 0$) of the L^2 metric if and only if the time-dependent vector field $u(x, t)$ defined by

$$\frac{d}{dt}\eta(x, t) = u \circ \eta(x, t) \tag{8}$$

is a solution of the Euler equations:

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla_u u + \nabla p &= 0 \\ \mathrm{div} u &= 0 \\ u(0) &= u_0 \end{aligned} \tag{9}$$

We have an L^2 Riemannian exponential map

$$\exp_e : \mathcal{U} \subseteq T_e\mathcal{D}^s_\mu(M) \to \mathcal{D}^s_\mu(M) \tag{10}$$

which can be viewed as follows: given a divergence-free vector field $u_0 \in \mathcal{U}$, let $u(x,t)$ be the unique solution of Euler equations with initial data u_0. Integrating this vector field, we get the position $\eta(x,t)$. Then,

$$\exp_e(u_0) = \eta(x,1)$$

It can be shown that this map is C^∞ smooth, and in fact a diffeomorphism in a neighborhood of $0 \in \mathcal{U}$.

This proves local well-posedness (in H^s) of the Euler equations in all dimensions.

Theorem (Ebin, Misiołek, Preston - 2006)

If $\dim(M) = 2$, $\exp_e$ *is a Fredholm map of index zero.*
If $\dim(M) = 3$, $\exp_e$ *is not Fredholm.*

Take $M = \mathbb{D}^2 \times \mathbb{S}^1$, and let $u_0 = \partial_\theta$. Then, then the image of

$$d(\exp_e)(\pi u_0) : T_e\mathcal{D}^s_\mu(M) \to T_{\exp_e(u_0)}\mathcal{D}^s_\mu(M)$$

is not closed.

Note that this is an axisymmetric flow, but not swirl-free.

Now let M be a Riemannian 3-manifold with a Killing vector field $K \neq 0$. Let $\{\phi_t\}$ be the flow of K. Consider the sets:

$$\begin{aligned} \mathcal{A}^s(M) &= \{\eta \in \mathcal{D}^s_\mu(M) : \eta \circ \phi_t = \phi_t \circ \eta, \ \forall t\} \\ T_e\mathcal{A}^s(M) &= \{u \in T_e\mathcal{D}^s_\mu(M) : [u, K] = 0\} \end{aligned} \tag{11}$$

Theorem 3 (L., Misiołek, Preston, 2018)

The set $\mathcal{A}^s(M)$ is a totally geodesic submanifold of $\mathcal{D}^s_\mu(M)$ with Lie algebra $T_e\mathcal{A}^s(M)$.

If u_0 is a vector field of sufficiently small swirl, then the map

$$d(\exp_e)(u_0) : T_e\mathcal{A}^s(M) \to T_{\exp_e(u_0)}\mathcal{A}^s(M) \tag{12}$$

is Fredholm of index zero.

Proof sketch: Let $\Phi(t) = t\, d(L_{\eta(t)^{-1}})\mathrm{d}(\exp_e)(tu_0)$. Use the group structure to obtain an integral equation for $\Phi(t)$:

$$\Phi(t) = \underbrace{\Omega(t)}_{\textit{invertible}} + \int_0^t \underbrace{\mathrm{Ad}^*_{\eta(t)}\mathrm{Ad}_{\eta(t)}\mathrm{ad}^*_{\Phi(\tau)}}_{\textit{bounded}}\, d\tau \tag{13}$$

where

$$\mathrm{ad}^*_{u_0} v = \begin{cases} \nabla\Delta^{-1}\langle v, \mathrm{curl}\, u_0\rangle & \text{if } \dim(M) = 2 \\ \mathrm{curl}\,\Delta^{-1}[v, \mathrm{curl}\, u_0] & \text{if } \dim(M) = 3 \end{cases} \tag{14}$$

When u_0 is swirl-free,

$$u_0 = a(r,z)\partial_r + b(r,z)\partial_z$$

then $\mathrm{curl}(u_0) = f(r,z)\partial_\theta$, so that

$$\begin{aligned} \mathrm{ad}^*_{u_0} v &= \mathrm{curl}\, \Delta^{-1}[v, \mathrm{curl}\, u_0] \\ &= \mathrm{curl}\, \Delta^{-1}\big(df(v)\partial_\theta\big) \end{aligned} \tag{15}$$

From here, the proof is finished in two steps:

- Operators map into correct spaces.
- Compactness of $\mathrm{ad}^* : H^s \to H^{s+1}$ via Rellich Lemma.

Fredholmness allows for a deeper understanding of singularities.

A vector $u_0 \in T_e\mathcal{D}^s_\mu(M)$ where $d(\exp_e)(u_0)$ is not invertible is called a *conjugate point*. The existence of conjugate points in $\mathcal{D}^s_\mu(M)$ was conjectured by Arnold and proved by Misiołek (1993). Many other examples were found since then:

- Shnirelman 1994; $\dim(M) \geq 3$;
- Misiołek 1996, when $M = \mathbb{T}^2$;
- Ebin, Misiołek, Preston 2006, $u = \partial_\theta$, $M = \mathbb{D}^2 \times \mathbb{S}^1$;
- Preston, Washabaugh 2014, $u = f(r)\partial_\theta$;
- Benn 2015, $\dim(M) = 2$, isometry group of M.

Assume that either $\dim(M) = 2$ or u_0 is swirl-free from now on.

We will focus on so-called **regular conjugate points**. This is an open and dense subset of all conjugate points.

The **multiplicity** or **order** of a conjugate point u_0 is the (finite) number $\dim \ker d \exp_e(u_0)$.

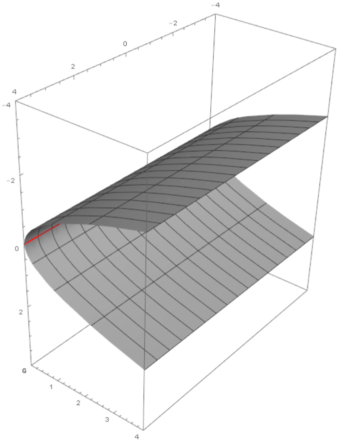
Theorem 4 (L. - 2018)

The set $C_e \subseteq T_e\mathcal{D}^s_\mu(M)$ of all regular conjugate points is a smooth submanifold of codimension 1. Its tangent space at any $u_0 \in C_e$ satisfies

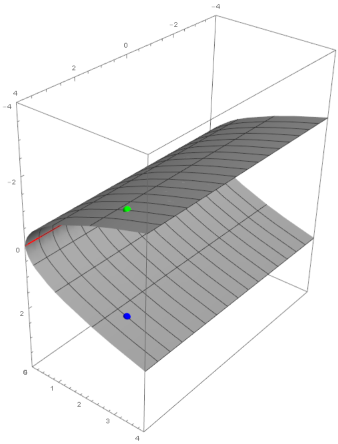
$$T_{u_0}C_e \oplus \mathbb{R}u_0 \simeq T_e\mathcal{D}^s_\mu(M).$$

Main ingredients:

- L^2 version of the Morse Index Theorem (Misiołek, Preston, 2009).
- Perturbation theory for self-adjoint operators.



Fold map: $(t, s) \mapsto (t^2, s)$



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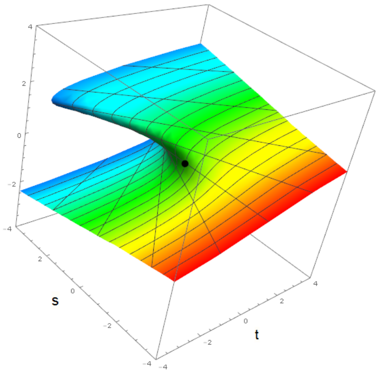
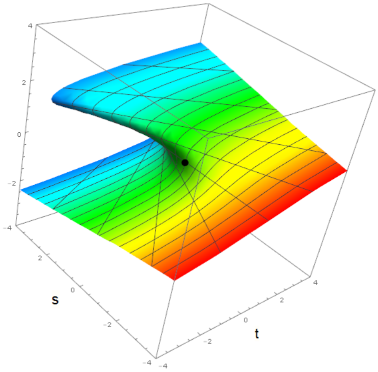


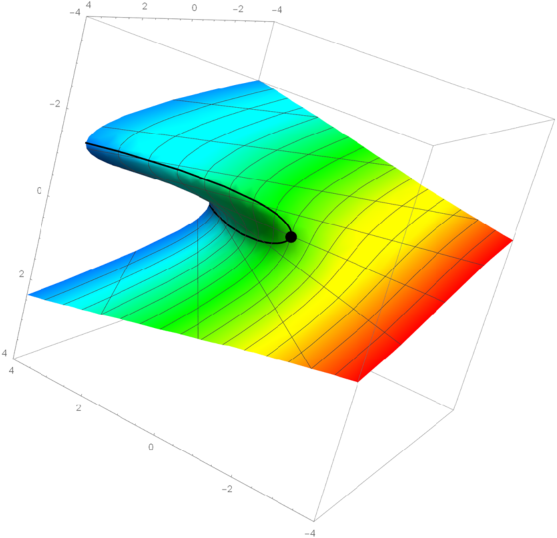
Cusp map: $(t, s) \mapsto (s, t^3 - st)$



$$Df(t,s) = \begin{pmatrix} 3t^2 - s & -t \\ 0 & 1 \end{pmatrix}$$

Singular set $= \{3t^2 = s\}$.

All points on the singular set are folds, except for $(0,0)$.



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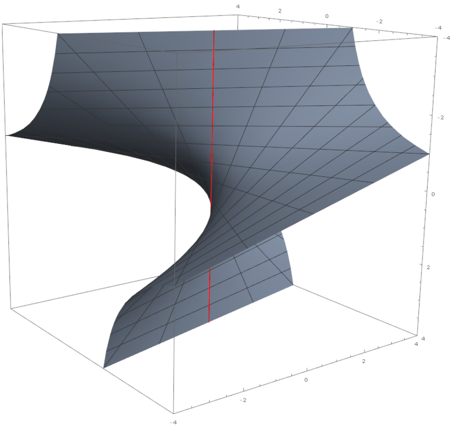
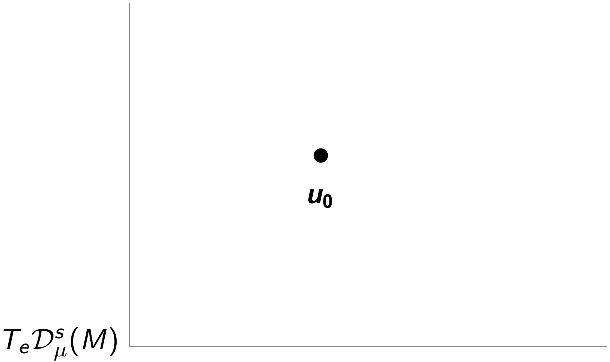
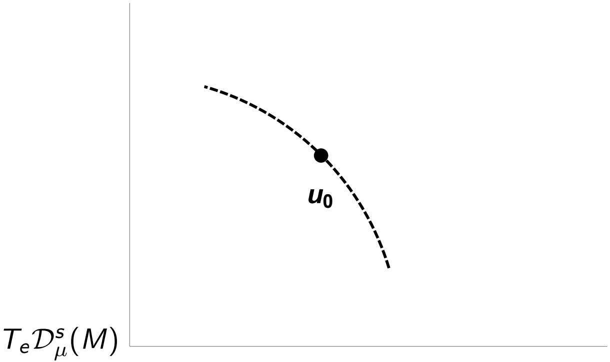


Map: $(t, x_1, \ldots, x_n) \mapsto (t, tx_1, \ldots, tx_n)$


u_0
$T_e\mathcal{D}^s_\mu(M)$


$T_e\mathcal{D}^s_\mu(M)$
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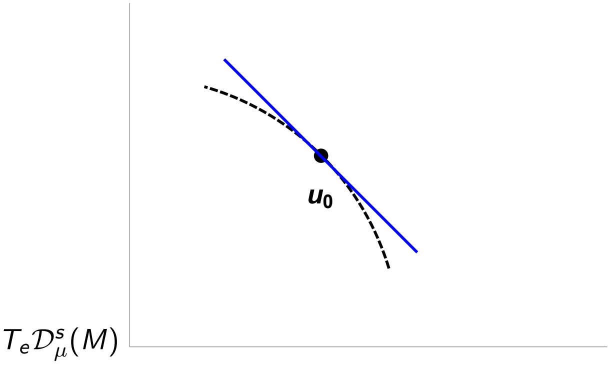
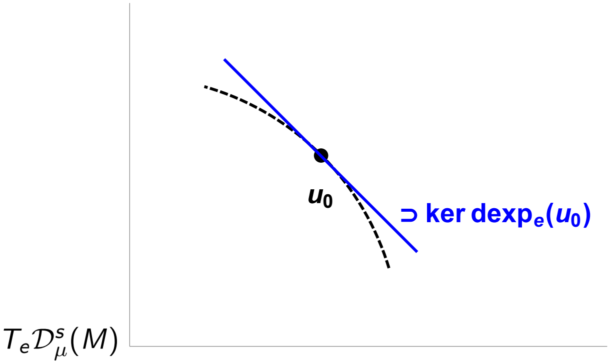
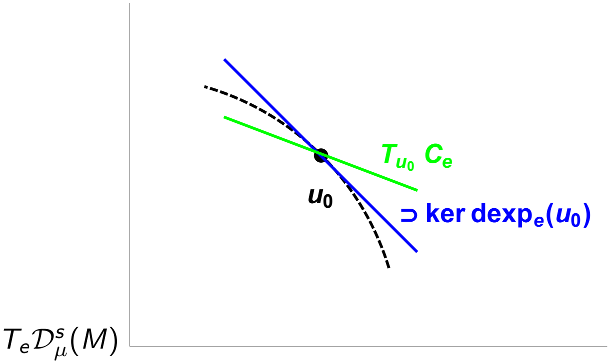


u_0
$T_e\mathcal{D}^s_\mu(M)$


u_0
$\supset \ker \operatorname{dexp}_e(u_0)$
$T_e\mathcal{D}^s_\mu(M)$



$T_{u_0} C_e$
u_0
$\supset \ker \mathrm{dexp}_e(u_0)$
$T_e\mathcal{D}^s_\mu(M)$

Theorem 5 (L. - 2018)

Let $u_0 \in C_e$ be a regular conjugate point of multiplicity 1 such that $\ker d\exp_e(u_0) \not\subseteq T_{u_0}C_e$. Then, in a neighborhood of u_0, $\exp_e$ has the normal form

$$\begin{aligned} \exp_e : \mathbb{R} \times \mathbb{H} &\to \mathbb{R} \times \mathbb{H} \\ (t, v) &\mapsto (t^2, v) \end{aligned}$$

Theorem 6 (L. - 2018)

Let $u_0 \in C_e$ be a regular conjugate point of multiplicity 1 such that $\ker d\exp_e(u_0) \subseteq T_{u_0} C_e$. Suppose u_0 is normal to C_e.
Let Π be the L^2 Weingarten tensor of $C_e \subseteq T_e \mathcal{D}^s_\mu(M)$. If

$$\Pi(w, w) \neq -\|w\|^2_{L^2}, \ \forall w \in \ker d\exp_e(u_0),$$

then near u_0, $\exp_e$ has the normal form

$$\begin{aligned} \exp_e : \mathbb{R}^2 \times \mathbb{H} &\to \mathbb{R}^2 \times \mathbb{H} \\ (t, s, v) &\mapsto (t^3 - st, s, v) \end{aligned}$$

Theorem 7 (L. - 2018)

Let $u_0 \in C_e$ be a regular conjugate point of multiplicity k such that $\ker d\exp_e(u) \subseteq T_u C_e$ for all u in a neighborhood of u_0. Then near u_0, $\exp_e$ has the normal form

$$\begin{aligned} \exp_e : \mathbb{R}^{k+1} \times \mathbb{H} &\to \mathbb{R}^{k+1} \times \mathbb{H} \\ (t, x_1, \ldots, x_k, v) &\mapsto (t, tx_1, tx_2, \ldots, tx_k, v) \end{aligned}$$

Remark 4

Every regular conjugate point of multiplicity $k \geq 2$ falls in this case.

Corollary 1 (L^2 Morse-Littauer)

The L^2 exponential map $\exp_e : T_e\mathcal{D}^s_\mu(M) \to \mathcal{D}^s_\mu(M)$ is not injective on any neighborhood of a conjugate point.

Proof. First, note that all of the above local forms are not injective. Let $u_0 \in T_e\mathcal{D}^s_\mu(M)$ be any regular conjugate point. One of the following holds:

- For all conjugate points u in a neighborhood of u_0, we have $\ker d\exp_e(u) \subseteq T_u C_e$.
- There exists a sequence $\{u_n\}_{n\geq 1}$ converging to u_0 with $\ker d\exp_e(u_n) \not\subseteq T_{u_n} C_e$.

- For all conjugate points u in a neighborhood of u_0, we have $\ker d\exp_e(u) \subseteq T_u C_e$.
- There exists a sequence $\{u_n\}_{n \geq 1}$ converging to u_0 with $\ker d\exp_e(u_n) \not\subseteq T_{u_n} C_e$.

In the first case, $\exp_e$ has a normal form at u_0, which is not injective.
In the second case, $\exp_e$ is a fold near each u_n, so it cannot be injective near u_0.

The result follows from the fact that regular conjugate points are dense in the set of all conjugate points. ■

Thank you!