Open problems

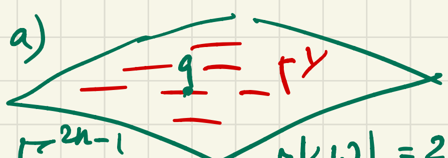
March 30, 2021

Problem 1: Darboux thm for sympl. mfds with corners

Recall: Darboux thm: Locally any sympl. mfd (M^{2n}, ω) is symplectomorphic to $(\mathbb{R}^{2n}, \sum dp_i \wedge dq_i)$ (i.e. the rank $2n$ is the only local invariant).

Givental's thm (local): Any $(\Gamma^k \subset M^{2n}, \omega)$ is defined up to a symplectomorphism by $\omega|_{\Gamma^k}$. Assuming that rank $\omega|_{\Gamma} = \text{const}$, it is the only local invariant

Ex: a)



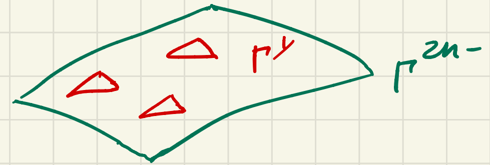
$M^{2n} \supset \Gamma^{2n-1}$ $\quad$ $\mathrm{rk}\,\omega|_{\Gamma} = 2n-1$

$$\Gamma^y_q = \{ \xi \in T_q M \mid \omega(\xi, u) = 0 \ \forall u \in T_q \Gamma \}$$

Ex: b) $\Gamma^{2n-2} \subset M$, 2 cases

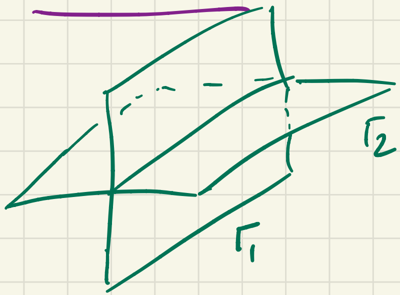
$\operatorname{rk} \omega|_{\Gamma} = 2n-2$

or



$\operatorname{rk} \omega|_{\Gamma} = 2n-4$

Problem A: Find a Darboux thm for a (normal) intersection $\Gamma_1 \cap \dots \cap \Gamma_k \subset (M^{2n}, \omega)$



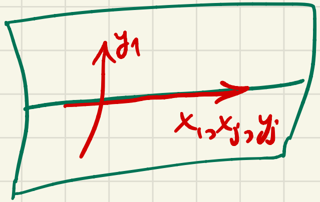
Conj: It is determined by $\omega|_{\Gamma_j}$, $\omega|_{\Gamma_i \cap \Gamma_j}$, ...

Rm: In dim 2, ω = vol form (Bruveris, Michor, ...)

Problem B: Describe relation to normal forms for log-symplectic (and b-sympl) mfds as an R-C-correspondence.

Namely, $\exists$ real notion of <u>b-sympl. mfds</u> (Guillemin, Miranda, Pires 2012)

Darboux thm: $\mathbb{R}^{2n}$, $\omega = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{j=2}^{n} dx_j \wedge dy_j$



Complex version: <u>log-symplectic mfds</u> (Deligne, K. Saito)

$$\mathbb{C}^{2n}, \quad \omega = \sum_{i,j=1}^{n} B_{ij} \frac{dy_i}{y_i} \wedge \frac{dy_j}{y_j} + \sum_{i=1}^{n} \frac{dy_i}{y_i} \wedge dz_i$$

<u>Rm</u>: $\exists$ real − complex correspondence (B.K. − Rosly)

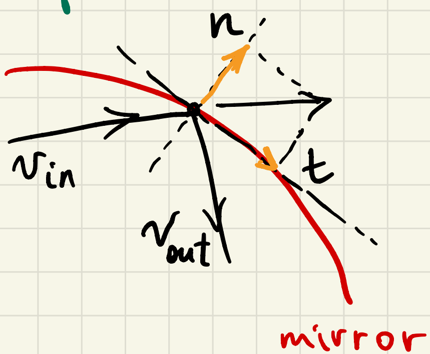
$\mathbb{R}$ $\quad\quad$ $\mathbb{C}$

a real manifold	$\leftrightarrow$	a complex manifold,
an orientation of the manifold	$\leftrightarrow$	a meromorphic volume form on the manifold,
manifold's boundary	$\leftrightarrow$	form's divisor of poles,
induced orientation of the boundary	$\leftrightarrow$	residue of the meromorphic form,
open manifold's infinity	$\leftrightarrow$	form's divisor of zeros,
Stokes formula	$\leftrightarrow$	Cauchy formula,
singular homology	$\leftrightarrow$	polar homology.

Problem 2: Describe Lorentz billiards at singular points in higher dimensions.

Consider $\mathbb{R}^{1+1}$ with a Lorentz metric, e.g. $dx^2 - dy^2$ or $dx\,dy$

Reflection law: $v_{in} = \vec{t} + \vec{n}$, $v_{out} = \vec{t} - \vec{n}$ holds for any metric



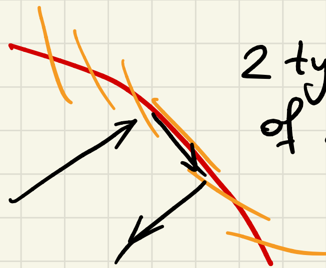
Singular points of a mirror

$\Leftrightarrow$ normal to a mirror is tangent to it

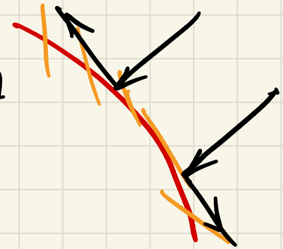
(it can happen in a Lorentz space)

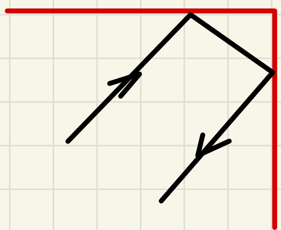
(B.K. - S. Tabachnikov 2009)

Ex:

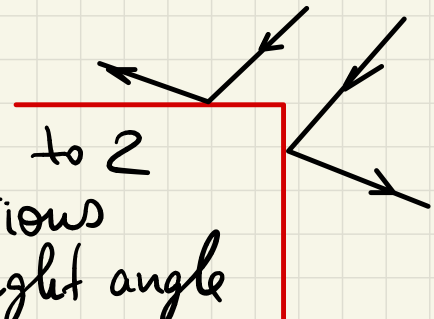


2 types of sing pt





similar to 2 reflections in a right angle

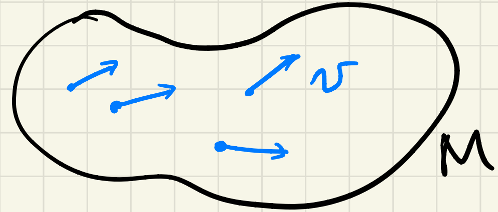


Problem: What happens in higher dimensions? Normal forms? Euclidean analogues?

Problem 3. Prove the non-integrability of the 2D Euler eq'n.

The Euler eq'n for an ideal fluid in a Riem. mfd M

$$\begin{cases} \partial_t v + \nabla_v v = -\nabla p \\ \operatorname{div}_\mu v = 0 \quad (\text{and } v \parallel \partial M) \end{cases}$$



v - velocity field of the fluid

In 2D the Euler eq'n is

$$\partial_t \omega = \{\psi, \omega\}, \text{ where } \quad v = \operatorname{sgrad} \psi, \quad \omega = \operatorname{curl} v = \Delta \psi$$

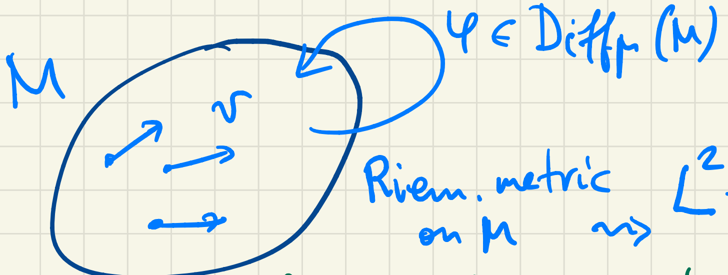
It has ∞ many conservation laws $I_k = \int \omega^k \mu$ (Casimirs)

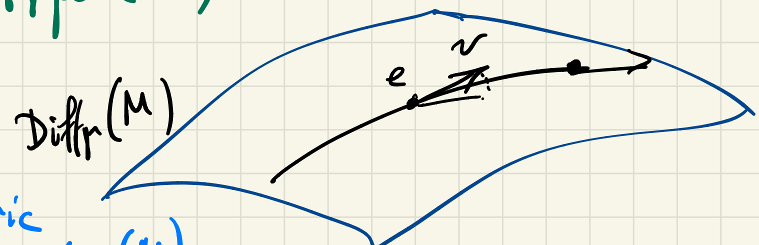
Problem: Prove that the 2D Euler eq'n is nonintegrable

(by any method: finite-dim approx, chaos, Ziglin, Morales-Ramis, Melnikov int'l on a submfd, ...)

Problem 4: Describe a variational principle for the Euler eq'n with sources and sinks

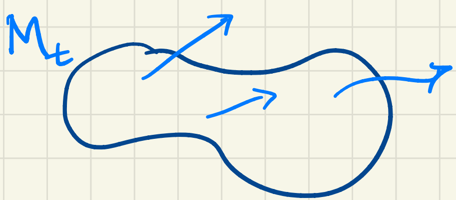
Recall: Arnold 1966: The Euler eq'n describes the geodesic flow on $\mathrm{Diff}_\mu(M)$

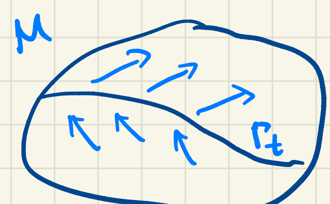




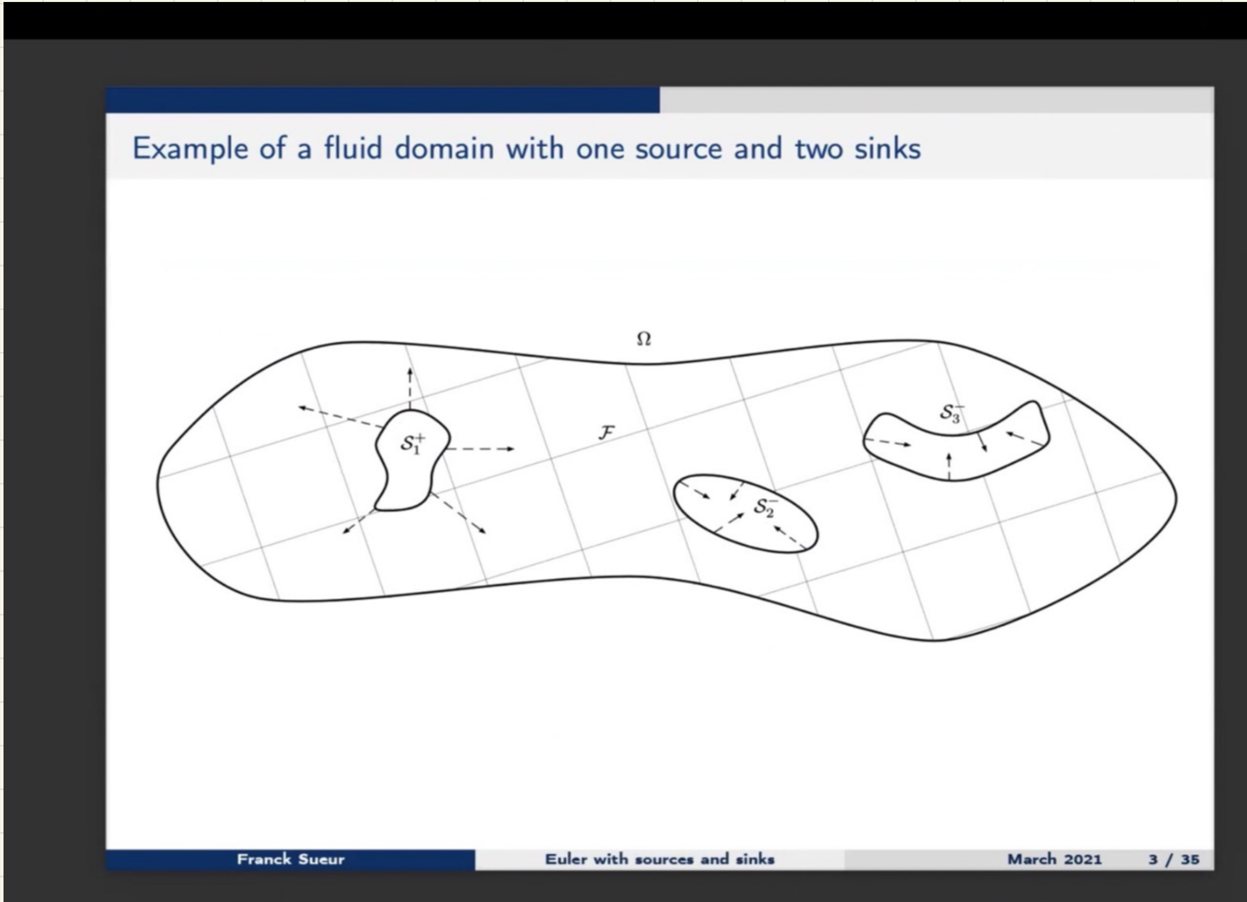
Riem. metric on M $\rightsquigarrow$ L^2-metric on $\mathrm{Diff}_\mu(M)$

Similarly, a fluid motion with dynamic boundary or vortex sheets corresponds to the geod. flows on groupoids (A. Izosimov - B.K. 2018)





In the talk by F. Sueur (Univ de Bordeaux) at CAM colloq Penn State:



Velocity formulation of Yudovich 1966's problem

The incompressible Euler equations read as :

$$
\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla p &= 0 && \text{in } \mathbb{R}^+ \times \mathcal{F}, \\
\operatorname{div} v &= 0 && \text{in } \mathbb{R}^+ \times \mathcal{F}, \\
v \cdot n &= g && \text{on } \mathbb{R}^+ \times \partial\mathcal{F}, \\
\operatorname{curl} v &= \omega^+ && \text{on } \mathbb{R}^+ \times \partial\mathcal{F}^+, \\
v(0,.) &= v_0 && \text{in } \mathcal{F},
\end{aligned}
$$

where

- $v : \mathbb{R}^+ \times \mathcal{F} \to \mathbb{R}^2$ is the fluid velocity field,
- $p : \mathbb{R}^+ \times \mathcal{F} \to \mathbb{R}$ is the fluid pressure,
- n is the unit normal vector field exiting from the domain $\mathcal{F}$,
- $g < 0$ at the inlet, $g > 0$ at the outlet, and $g = 0$ on $\mathbb{R}^+ \times \partial\Omega$. Moreover, at any time t, the function $g(t, \cdot)$ has zero average on $\partial\mathcal{F}$,
- $\omega^+ : \mathbb{R}^+ \times \partial\mathcal{F}^+$ is the entering vorticity,
- the initial data v_0 is assumed to satisfy $\operatorname{div} v_0 = 0$ in $\mathcal{F}$.

- The vorticity $\omega := \operatorname{curl} v$ satisfies the transport equation :

$$\begin{aligned} \partial_t \omega + v \cdot \nabla \omega &= 0 && \text{in } \mathbb{R}^+ \times \mathcal{F}, \\ \omega &= \omega^+ && \text{on } \mathbb{R}^+ \times \partial\mathcal{F}^+, \\ \omega(0,.) &= \omega_0 && \text{in } \mathcal{F}. \end{aligned}$$

The velocity v can be recovered from ω by :

$$\begin{aligned} \operatorname{div} v &= 0 && \text{in } \mathcal{F}, \\ \operatorname{curl} v &= \omega && \text{in } \mathcal{F}, \\ v \cdot n &= g && \text{on } \partial\mathcal{F}, \\ \int_{\partial\mathcal{S}^i} v(t,.) \cdot \tau &= \mathcal{C}_i(t) && \text{for } i \in \mathcal{I}, \end{aligned}$$

where

- τ is the counterclockwise tangent vector to the boundary.,
- the $\mathcal{C}_i(t)$ are the circulations of the velocity vector field v around the connected components $\partial\mathcal{S}^i$.

This framework allowed one to prove various existence, uniqueness, and stability results.

Problem: What variational principle is responsible for the Euler eq'n with sources and sinks?

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